

UNIQUENESS OF OPTIMAL SOLUTIONS FOR SEMI-DISCRETE TRANSPORT WITH p -NORM COST FUNCTIONS

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ABSTRACT. Semi-discrete transport can be characterized in terms of real-valued shifts. Often, but not always, the solution to the shift-characterized problem partitions the continuous region. This paper gives examples of when partitioning fails, and offers a large class of semi-discrete transport problems where the shift-characterized solution is always a partition.

1. INTRODUCTION

Optimal transport offers a way to measure the distance between two probability spaces, X and Y . In the class of transport problems known as semi-discrete optimal transport, the probability distribution on X is almost-everywhere continuous and the probability distribution on Y is discrete, with N points of positive measure. Given minimal assumptions, described below, the semi-discrete problem always has at least one solution that partitions X into N regions based on transport destination.

Rüschendorf and Uckelmann developed a way to characterize semi-discrete transport in terms of a set of real-valued shifts. This shift characterization often results in a solution that partitions X into N regions. Unfortunately, the shift characterization does not always partition X . This important fact has not always been recognized or clearly expressed in the literature; see [5, 6, 7]. To remedy that ambiguity, this paper gives clear, specific examples where shift-characterized partitioning fails, and it offers a large class of problems where the shift characterization is guaranteed to partition X .

2. BACKGROUND

2.1. General optimal transport: the Monge-Kantorovich and Monge problems. Though this paper focuses on the semi-discrete problem, it is worth describing it in terms of the more general, Monge-Kantorovich transport problem.

Definition 2.1 (Monge-Kantorovich problem). Let $X, Y \subseteq \mathbb{R}^d$, let μ and ν be probability densities defined on X and Y , and let $c(\mathbf{x}, \mathbf{y}) : X \times Y \rightarrow \mathbb{R}$ be a continuous measurable *ground cost* function. Define the set of *transport plans*

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X \times Y) \mid \begin{array}{l} \pi[A \times Y] = \mu[A], \pi[X \times B] = \nu[B], \\ \forall \text{ meas. } A \subseteq X, B \subseteq Y \end{array} \right\}, \quad (2.1)$$

where $\mathcal{P}(X \times Y)$ is the set of probability measures on the product space, and define the *primal cost* function $P : \Pi(\mu, \nu) \rightarrow \mathbb{R}$ as

$$P(\pi) := \int_{X \times Y} c(\mathbf{x}, \mathbf{y}) d\pi(\mathbf{x}, \mathbf{y}). \quad (2.2)$$

The Monge-Kantorovich problem is to find the *optimal primal cost*

$$P^* := \inf_{\pi \in \Pi(\mu, \nu)} P(\pi), \quad (2.3)$$

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and an associated *optimal transport plan*

$$\pi^* := \arg \inf_{\pi \in \Pi(\mu, \nu)} P(\pi). \quad (2.4)$$

Under the conditions given, an optimal transport plan, π^* , is guaranteed to exist. However, π^* may not be unique, or even a.e.-unique. Furthermore, the existence of π^* , an optimal *plan*, does not ensure that π^* is a *map*, or that an optimal map exists. Nonetheless, consider the form such an optimal map would take.

Definition 2.2 (Monge problem). In certain cases, there exists at least one solution to the semi-discrete Monge-Kantorovich problem that does not split transported masses. In other words, there exists some π^* such that

$$\pi^*(\mathbf{x}, \mathbf{y}) = \pi_{T^*}^*(\mathbf{x}, \mathbf{y}) := \mu(\mathbf{x}) \chi[\mathbf{y} = T^*(\mathbf{x})], \quad (2.5)$$

where $T^* : X \rightarrow Y$ is a measurable map called the *optimal transport map*. When such a π^* exists, we say the solution also solves the Monge problem.

If the Monge problem has a solution, we can assume without loss of generality that every $\pi \in \Pi(\mu, \nu)$ satisfies

$$\pi(\mathbf{x}, \mathbf{y}) = \pi_T(\mathbf{x}, \mathbf{y}) := \mu(\mathbf{x}) \chi[\mathbf{y} = T(\mathbf{x})], \quad (2.6)$$

for some measurable transport map $T : X \rightarrow Y$, and that the primal cost can be written

$$P(\pi) := \int_X c(\mathbf{x}, T(\mathbf{x})) d\mu(\mathbf{x}). \quad (2.7)$$

2.2. Semi-discrete optimal transport and the shift characterization. The semi-discrete optimal transport problem is the Monge-Kantorovich problem of Theorem 2.1, with restrictions on μ and ν :

- (1) Assume that μ satisfies the following:
 - (a) μ is bounded.
 - (b) μ is nonatomic.
 - (c) μ is continuous except on a set of Lebesgue measure zero.
 - (d) The support of μ is contained in the convex compact region $A \subseteq X$.
- (2) Assume ν has exactly $n \geq 2$ non-zero values, located at $\{\mathbf{y}_i\}_{i=1}^n \subseteq Y$.

Because c is continuous and μ is nonatomic, at least one solution to the semi-discrete Monge-Kantorovich problem also satisfies the Monge problem, described in Theorem 2.2; see [4]. Thus, by applying Equation (2.6), we can assume without loss of generality that any transport plan π has an associated map T , and that T partitions A into n sets A_i , where A_i is the set of points in A that are transported by T to \mathbf{y}_i . Using this partitioning scheme in combination with Equation (2.7) allows us to rewrite the primal cost function for the semi-discrete problem as

$$P(\pi) := \sum_{i=1}^n \int_{A_i} c(\mathbf{x}, \mathbf{y}_i) d\mu(\mathbf{x}). \quad (2.8)$$

This idea of sets A_i is central to describing the shift characterization of the semi-discrete optimal transport problem. The following definition is based on one given by Rüschendorf and Uckelmann in [5, 7].

Definition 2.3 (Shift characterization). Let $\{a_i\}_{i=1}^n$ be a set of n finite values, referred to as *shifts*. Define

$$F(\mathbf{x}) := \max_{1 \leq i \leq n} \{a_i - c(\mathbf{x}, \mathbf{y}_i)\}. \quad (2.9)$$

For $i \in \mathbb{N}_n$, where $\mathbb{N}_n = \{1, \dots, n\}$, let

$$A_i := \{\mathbf{x} \in A \mid F(\mathbf{x}) = a_i - c(\mathbf{x}, \mathbf{y}_i)\}. \quad (2.10)$$

Note that $\cup_{i=1}^n A_i = A$. The problem of determining an optimal transport plan π^* is equivalent to determining shifts $\{a_i\}_{i=1}^n$ such that for all $i \in \mathbb{N}_n$, the total mass transported from A_i to \mathbf{y}_i equals $\nu(\mathbf{y}_i)$.

2.3. Formalizing the shift-characterized partition. “Partitioning” A is described in [7, 5] as $\mu(A_i) = \nu(\mathbf{y}_i)$. However, it is beneficial to describe the shift-characterized partition in more detail. Doing so requires a few additional definitions.

Definition 2.4 (Boundaries and boundary sets). For all $i, j \in \mathbb{N}_n$ such that $i \neq j$, let

$$A_{ij} := A_i \cap A_j. \quad (2.11)$$

The *boundary set* is defined as

$$B := \bigcup_{1 \leq i < n} \bigcup_{i < j \leq n} A_{ij}. \quad (2.12)$$

For all $i, j \in \mathbb{N}_n$ such that $i \neq j$, define $g_{ij} : X \rightarrow \mathbb{R}$ as

$$g_{ij}(\mathbf{x}) := c(\mathbf{x}, \mathbf{y}_i) - c(\mathbf{x}, \mathbf{y}_j). \quad (2.13)$$

Definition 2.5 (F μ -partitions A). Let F be as defined in Equation (2.9), and the sets A_i as defined in Equation (2.10) for $i \in \mathbb{N}_n$. Then one says F μ -partitions the set A , or F is called a μ -partition, if

- (1) $\mu(A) < \infty$,
- (2) for all $i, j \in \mathbb{N}_n$, $i \neq j$, $\mu(A_{ij}) = 0$,
- (3) $\sum_{i=1}^n \mu(A_i) = \mu(A)$, and
- (4) for all $i \in \mathbb{N}_n$, $\mu(A_i) = \nu(\mathbf{y}_i) > 0$.

Definition 2.6 (Monge under the shift characterization). We say a transport plan π is *Monge under the shift characterization* if π has an associated transport map T , a function F , as described in Equation (2.9), and sets $\{A_i\}_{i=1}^n$, as described in Equation (2.10), such that for all $\mathbf{x} \in A$,

$$\mathbf{x} \in \mathring{A}_i \text{ for some } i \in \mathbb{N}_n \quad \implies \quad T(\mathbf{x}) = \mathbf{y}_i. \quad (2.14)$$

In other words, F μ -partitions A and T agrees with F on $A \setminus B$.

If $\mu(B) > 0$ for the shifts $\{a_i\}_{i=1}^n$, no such transport plan π can exist, and the transport problem itself can be said to be *not Monge under the shift characterization*. Conversely, if $\mu(B) = 0$, then such a transport plan exists, and so the transport problem itself is said to be Monge under the shift characterization. In other words, F μ -partitions A if and only if the transport problem is Monge under the shift characterization.

The following result, from [3], allows us to go further:

Theorem 2.7. *Suppose one has a semi-discrete transport problem, as described in Section 2.2. Let F be as defined in Equation (2.9), and the sets A_i as defined in Equation (2.10) for $i \in \mathbb{N}_n$. Then F μ -partitions A if and only if $\mu(B) = 0$.*

Taken together, these statements provide a formal definition and condition for what it means for the shift-characterized solution to partition A :

The shift-characterized semi-discrete transport problem partitions A — that is, F μ -partitions A — if and only if the semi-discrete transport problem is Monge under the shift characterization, which is true if and only if $\mu(B) = 0$.

2.4. Uniqueness of semi-discrete transport solutions. Given the semi-discrete transport problem described in Section 2.2, Corollary 4 of [2] provides a sufficient condition for the existence of a Monge solution that is unique μ -a.e.:

$$\mu(\{\mathbf{x} \in A \mid c(\mathbf{x}, \mathbf{y}_i) - c(\mathbf{x}, \mathbf{y}_j) = k\}) = 0 \quad \forall i, j \in \mathbb{N}_n, i \neq j, \quad \forall k \in \mathbb{R}. \quad (2.15)$$

If Equation (2.15) is satisfied, $\mu(B) = 0$. Therefore, if Equation (2.15), then the transport problem is Monge under the shift characterization and the transport solution is unique μ -a.e.

However, if a transport problem is Monge under the shift characterization, then it has a unique μ -a.e. shift-characterized solution, whether or not Equation (2.15) is satisfied. This statement is formalized and proved in [3] as the following theorem:

Theorem 2.8 (The optimal transport map is unique μ -a.e.). *Given a semi-discrete transport problem, let π^* and $\tilde{\pi}^*$ be optimal transport plans that are both Monge under the shift characterization. If T is a transport map associated with π^* , and \tilde{T} a transport map associated with $\tilde{\pi}^*$, then $T = \tilde{T}$ except on a set of μ -measure zero.*

3. MATHEMATICAL SUPPORT

While Equation (2.15) implies $\mu(B) = 0$, the converse is not true, as Section 3.1 shows. Next, Section 3.2 identifies a large class of problems where both conditions hold and the solution is always unique μ -a.e.

3.1. Partitioning with the 1-norm and ∞ -norm. Let $X = [0, 1]^2$, $Y = \{\mathbf{y}_1, \mathbf{y}_2\}$, and let μ be the continuous uniform distribution. This simple setup can be used to demonstrate failure to partition for both the uniform norm (∞ -norm) and the Manhattan norm (1-norm).

3.1.1. The uniform norm. Let $\mathbf{y}_1 = (1/4, 1/2)$ and $\mathbf{y}_2 = (3/4, 1/2)$, and let $c : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be the uniform norm (∞ -norm): $c(\mathbf{x}, \mathbf{y}) = \max_{i \in \{1, 2\}} |x_i - y_i|$ for all $\mathbf{x} = (x_1, x_2) \in X$, $\mathbf{y} = (y_1, y_2) \in Y$. Consider two examples:

- (1) If $\nu(\mathbf{y}_1) = 1/32$, then $\nu(\mathbf{y}_2) = 31/32$. In this case, $\mu(B) = 1/16$, and the shift-characterized solution fails to partition A . See Figure 1(a).
- (2) However, if $\nu(\mathbf{y}_1) = 1/8$, then $\nu(\mathbf{y}_2) = 7/8$, $\mu(B) = 0$ and the shift-characterized solution does partition A . See Figure 1(b).

Even though one of the problems illustrated in Figure 1 results in a partition, Equation (2.15) fails in both cases:

$$\mu(\{\mathbf{x} \in A \mid c(\mathbf{x}, \mathbf{y}_2) - c(\mathbf{x}, \mathbf{y}_1) = k\}) = \frac{1}{16} \quad \text{if} \quad k \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}.$$

In general, for this choice of X , Y , μ , and c , the shift-characterized solution partitions A if and only if

$$\nu(\mathbf{y}_1) \in \left(\frac{1}{16}, \frac{15}{16}\right).$$

Thus, when c is the uniform norm, one can have $\mu(B) = 0$, giving a shift-characterized partition of A that is unique μ -a.e., whether or not Equation (2.15) is satisfied.

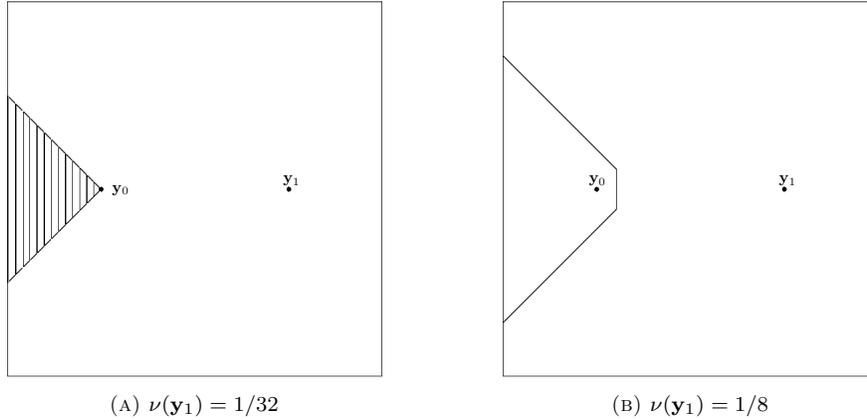


FIGURE 1. ∞ -norm partitioning example

3.1.2. The Manhattan norm. Now let $\mathbf{y}_1 = (1/4, 1/4)$ and $\mathbf{y}_2 = (3/4, 3/4)$. Let $c : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be the Manhattan norm (1-norm): $c(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$ for all $\mathbf{x} = (x_1, x_2) \in X$, $\mathbf{y} = (y_1, y_2) \in Y$. Consider three examples:

- (1) If $\nu(\mathbf{y}_1) = 1/2$, then $\nu(\mathbf{y}_2) = 1/2$. In this case, $\mu(B) = 1/8$, and the shift-characterized solution fails to partition A . See Figure 2(a) for an illustration.
- (2) If $\nu(\mathbf{y}_1) = 1/32$, then $\nu(\mathbf{y}_2) = 31/32$ and $\mu(B) = 1/8$, so the shift-characterized solution again fails to partition A . This is shown in Figure 2(b).
- (3) However, if $\nu(\mathbf{y}_1) = 1/4$, then $\nu(\mathbf{y}_2) = 3/4$. In this case $\mu(B) = 0$ and the shift-characterized solution does partition A . See Figure 2(c).

Once again, Equation (2.15) fails in all the Figure 2 cases:

$$\mu(\{\mathbf{x} \in A \mid c(\mathbf{x}, \mathbf{y}_2) - c(\mathbf{x}, \mathbf{y}_1) = k\}) = \frac{1}{8} \quad \text{if} \quad k \in \left\{-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right\}.$$

In fact, for this choice of X , Y , μ , and c , the shift-characterized solution partitions A if and only if

$$\nu(\mathbf{y}_1) \in \left(\frac{1}{16}, \frac{7}{16}\right) \cup \left(\frac{9}{16}, \frac{15}{16}\right).$$

Thus, as Figure 2 illustrates, when c is the 1-norm, one can have $\mu(B) = 0$, giving a shift-characterized partition of A that is unique μ -a.e., whether or not Equation (2.15) is satisfied.

Figure 2(a) is worth special consideration, because it is not simply a non-partitioning shift-characterized transport solution: it also constitutes a failed Voronoi diagram. One can see a similar example in Figure 37 of [1], offered as part of a discussion on methods for resolving lack of partitioning and uniqueness for certain Voronoi diagrams.

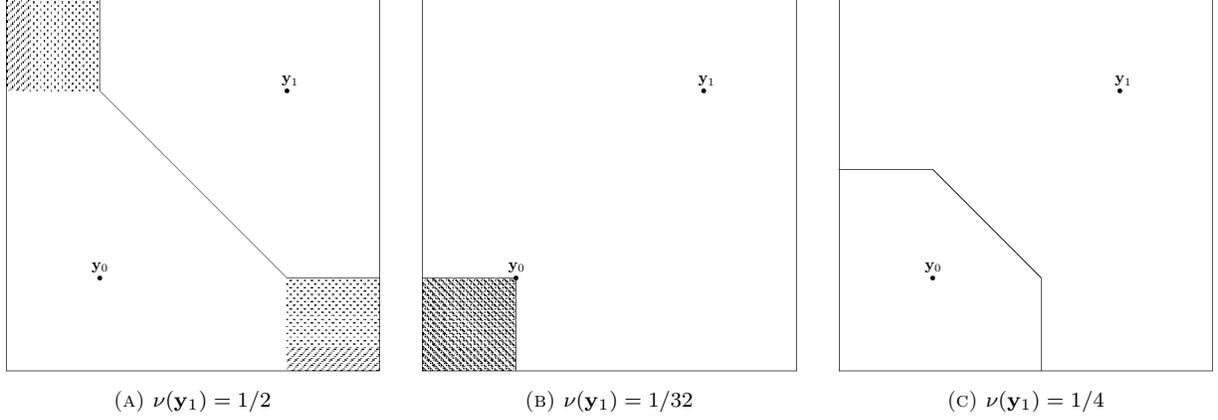


FIGURE 2. 1-norm partitioning example

3.2. Partitioning with p -norms when $p \in (1, \infty)$ -norm. Given the semi-discrete transport assumptions already stated, let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a p -norm with $p \in (1, \infty)$:

$$c(\mathbf{x}, \mathbf{y}) := \left[\sum_{i=1}^d |x_i - y_i|^p \right]^{1/p}, \quad \forall \mathbf{x} = (x_1, \dots, x_d) \in X, \quad \forall \mathbf{y} = (y_1, \dots, y_d) \in Y. \quad (3.1)$$

Then the semi-discrete transport problem is always Monge under the shift characterization.

This assertion will be shown in two steps:

- (1) If g_{ij} , defined in Equation (2.13), is equal to the constant value $a_i - a_j$ in some neighborhood of $\mathbf{x}_0 \in A_{ij}$, then $|a_i - a_j| = c(\mathbf{y}_i, \mathbf{y}_j)$. [Theorem 3.1]
- (2) It follows from Step (1) that $\mu(B) > 0$ implies the existence of a ball of positive radius whose points are all collinear with both \mathbf{y}_i and \mathbf{y}_j . [Theorem 3.2]

Because of the contradiction inherent in Step (2), $\mu(B) = 0$, and so Theorem 3.2 concludes that the problem must be Monge under the shift characterization.

Theorem 3.1. *Let c be a p -norm with $p \in (1, \infty)$, and $\mathbf{x}_0 \in A_{ij}$ for some $i, j \in \mathbb{N}_n$, $i \neq j$. If $g_{ij}(\mathbf{x}) = a_i - a_j$ for all \mathbf{x} in a neighborhood of \mathbf{x}_0 , then $|a_i - a_j| = c(\mathbf{y}_i, \mathbf{y}_j)$.*

Proof. Let c be a p -norm with $p \in (1, \infty)$, $\mathbf{x}_0 \in A_{ij}$, and $g_{ij}(\mathbf{x}) = a_i - a_j$ for all \mathbf{x} in some neighborhood of \mathbf{x}_0 . Suppose to the contrary, however, that $|a_i - a_j| \neq c(\mathbf{y}_i, \mathbf{y}_j)$.

Say $|a_i - a_j| > c(\mathbf{y}_i, \mathbf{y}_j)$, and assume without loss of generality that $|a_i - a_j| = a_i - a_j$. Then

$$g_{ij}(\mathbf{x}_0) = c(\mathbf{x}_0, \mathbf{y}_i) - c(\mathbf{x}_0, \mathbf{y}_j) = a_i - a_j > c(\mathbf{y}_i, \mathbf{y}_j),$$

which implies $c(\mathbf{x}_0, \mathbf{y}_i) > c(\mathbf{x}_0, \mathbf{y}_j) + c(\mathbf{y}_i, \mathbf{y}_j)$. This is a violation of the triangle inequality. Therefore, it must be the case that $|a_i - a_j| < c(\mathbf{y}_i, \mathbf{y}_j)$.

For all $k \in \mathbb{N}_n$, define $c_k(\mathbf{x}) := c(\mathbf{x}, \mathbf{y}_k)$. Because $|a_i - a_j| < c(\mathbf{y}_i, \mathbf{y}_j)$, $\mathbf{x}_0 \neq \mathbf{y}_i$ and $\mathbf{x}_0 \neq \mathbf{y}_j$. Hence, $c_i(\mathbf{x}_0) > 0$ and $c_j(\mathbf{x}_0) > 0$.

Because g_{ij} is constant in a neighborhood of \mathbf{x}_0 , $\nabla g_{ij}(\mathbf{x}_0) = \nabla c_i(\mathbf{x}_0) - \nabla c_j(\mathbf{x}_0) = 0$, which implies $\nabla c_i(\mathbf{x}_0) = \nabla c_j(\mathbf{x}_0)$. Hence, each of the first-order partial derivatives of c_i and c_j are equal at \mathbf{x}_0 .

Assume $\mathbf{x}_0 = (x_1, \dots, x_d)$, $\mathbf{y}_i = (y_1^i, \dots, y_d^i)$, and $\mathbf{y}_j = (y_1^j, \dots, y_d^j)$. Then the equality of the k -th partial derivatives, $\nabla_{x_k} c_i(\mathbf{x}_0) = \nabla_{x_k} c_j(\mathbf{x}_0)$, gives

$$(x_k - y_k^i) |x_k - y_k^i|^{p-2} (c_i(\mathbf{x}_0))^{1-p} = (x_k - y_k^j) |x_k - y_k^j|^{p-2} (c_j(\mathbf{x}_0))^{1-p}.$$

Thus, $x_k - y_k^i$ and $x_k - y_k^j$ have the same sign or are both zero. Because $p > 1$, $p - 1 > 0$. Hence, taking the $(p - 1)$ -th root of both sides,

$$\frac{x_k - y_k^i}{c_i(\mathbf{x}_0)} = \frac{x_k - y_k^j}{c_j(\mathbf{x}_0)} \quad \forall k \in \mathbb{N}_d. \quad (3.2)$$

As a consequence of Equation (3.2), $x_k - y_k^i = 0$ if and only if $x_k - y_k^j = 0$. Hence, $x_k = y_k^i$ if and only if $x_k = y_k^j$.

Let K be the total number of k -th directional components satisfying $x_k \neq y_k^i$. Consider three cases: $K = 0$, $K = 1$, and $K > 1$.

$K = 0$: Then $x_k = y_k^i = y_k^j$ for all $k \in \mathbb{N}_d$, in which case $\mathbf{y}_i = \mathbf{y}_j$. Since the semi-discrete transport problem requires distinct non-zero points in Y , it must be the case that $i = j$, contradicting the initial assumption that $i \neq j$. Hence, $K \geq 1$.

$K = 1$: There exists exactly one k such that the components are not equal. Since $x_k - y_k^i$ and $x_k - y_k^j$ have the same sign,

$$|g_{ij}(\mathbf{x}_0)| = |(x_k - y_k^i) - (x_k - y_k^j)| = |y_k^j - y_k^i| = c(\mathbf{y}_i, \mathbf{y}_j).$$

This contradicts the assumption that $|a_i - a_j| < c(\mathbf{y}_i, \mathbf{y}_j)$, and hence $K > 1$.

$K > 1$: Because g_{ij} is constant in some neighborhood of \mathbf{x}_0 , it must also be the case that $\nabla^2 g_{ij}(\mathbf{x}_0) = 0$. Hence, $\nabla^2 c_i(\mathbf{x}_0) = \nabla^2 c_j(\mathbf{x}_0)$, so each of the second-order partial derivatives of c_i and c_j are equal at \mathbf{x}_0 . The equality of the second-order partial derivatives taken with respect to x_k gives

$$\frac{(p-1)|x_k - y_k^i|^{p-2}}{(c_i(\mathbf{x}_0))^{2p-1}} [(c_i(\mathbf{x}_0))^p - |x_k - y_k^i|^p] = \frac{(p-1)|x_k - y_k^j|^{p-2}}{(c_j(\mathbf{x}_0))^{2p-1}} [(c_j(\mathbf{x}_0))^p - |x_k - y_k^j|^p], \quad (3.3)$$

which can be rewritten as

$$\frac{p-1}{c_i(\mathbf{x}_0)} \left(\frac{|x_k - y_k^i|}{c_i(\mathbf{x}_0)} \right)^{p-2} \left[1 - \left(\frac{|x_k - y_k^i|}{c_i(\mathbf{x}_0)} \right)^p \right] = \frac{p-1}{c_j(\mathbf{x}_0)} \left(\frac{|x_k - y_k^j|}{c_j(\mathbf{x}_0)} \right)^{p-2} \left[1 - \left(\frac{|x_k - y_k^j|}{c_j(\mathbf{x}_0)} \right)^p \right]. \quad (3.4)$$

Applying Equation (3.2), define

$$\sigma_k = \frac{|x_k - y_k^i|}{c_i(\mathbf{x}_0)} = \frac{|x_k - y_k^j|}{c_j(\mathbf{x}_0)}.$$

Then Equation (3.4) can be rewritten as

$$\frac{p-1}{c_i(\mathbf{x}_0)} \sigma_k^{p-2} (1 - \sigma_k^p) = \frac{p-1}{c_j(\mathbf{x}_0)} \sigma_k^{p-2} (1 - \sigma_k^p). \quad (3.5)$$

By assumption, for all $k \in \mathbb{N}_d$, $x_k - y_k^i \neq 0$ and $x_k - y_k^j \neq 0$. Hence, $\sigma_k > 0$.

Since $d > 1$, and $|x_k - y_k^i| > 0$ for all $k \in \mathbb{N}_d$, it must be that $|x_k - y_k^i| < c_i(\mathbf{x}_0)$ for all $k \in \mathbb{N}_d$. Therefore,

$$\sigma_k = \frac{|x_k - y_k^i|}{c_i(\mathbf{x}_0)} < 1,$$

which implies $1 - \sigma_k^p > 0$. Therefore, $(p-1)\sigma_k^{p-2}(1 - \sigma_k^p) > 0$, and Equation (3.5) simplifies to $\frac{1}{c_i(\mathbf{x}_0)} = \frac{1}{c_j(\mathbf{x}_0)}$.

Thus, $c_i(\mathbf{x}_0) = c_j(\mathbf{x}_0)$. Combining this with Equation (3.2) implies $y_k^i = y_k^j$ for all $k \in \mathbb{N}_d$, and so $\mathbf{y}_i = \mathbf{y}_j$. Since $\mathbf{y}_i = \mathbf{y}_j$, and the semi-discrete transport problem requires distinct non-zero points in Y , it must be the case that $i = j$, contradicting the initial assumption that $i \neq j$. Thus, $K \not\geq 1$.

All choices of K lead to contradictions. Hence, if c is a p -norm for some $p \in (1, \infty)$, $\mathbf{x}_0 \in A_{ij}$, $i \neq j$, and $g_{ij}(\mathbf{x}) = a_i - a_j$ for all \mathbf{x} in some neighborhood of $\mathbf{x}_0 \in A_{ij}$, then it must be the case that $|a_i - a_j| = c(\mathbf{y}_i, \mathbf{y}_j)$. \square

Theorem 3.2. *If c is a p -norm for some $p \in (1, \infty)$, then the semi-discrete transport problem is Monge under the shift characterization.*

Proof. Assume the contrary is true. Then $\mu(B) > 0$, so $\mu(A_{ij}) > 0$ for some $i, j \in \mathbb{N}_n, i \neq j$. Because μ is nonatomic, there exist $\mathbf{x}_0 \in A_{ij}$ and $\epsilon > 0$ such that the ball $\mathcal{B}_\epsilon(\mathbf{x}_0)$, defined with respect to the Euclidean space \mathbb{R}^d , satisfies $\mathcal{B}_\epsilon(\mathbf{x}_0) \subseteq A_{ij}$ and $\mu(\mathcal{B}_\epsilon(\mathbf{x}_0)) > 0$. By Theorem 3.1, $|a_i - a_j| = c(\mathbf{y}_i, \mathbf{y}_j)$. Assume without loss of generality that $|a_i - a_j| = a_i - a_j$.

Let $\mathbf{x} \in \mathcal{B}_\epsilon(\mathbf{x}_0)$. Since $\mathbf{x} \in A_{ij}$,

$$g_{ij}(\mathbf{x}) = a_i - a_j \iff c(\mathbf{x}, \mathbf{y}_i) - c(\mathbf{x}, \mathbf{y}_j) = c(\mathbf{y}_i, \mathbf{y}_j) \iff c(\mathbf{x}, \mathbf{y}_i) = c(\mathbf{x}, \mathbf{y}_j) + c(\mathbf{y}_i, \mathbf{y}_j).$$

Because c is a p -norm and $p \in (1, \infty)$, Minkowski's inequality implies that \mathbf{x} , \mathbf{y}_i , and \mathbf{y}_j are all collinear. The choice of \mathbf{x} was nonspecific, and therefore every point in the ball $\mathcal{B}_\epsilon(\mathbf{x}_0)$ must be collinear with the points \mathbf{y}_i and \mathbf{y}_j .

Of course, this is impossible, and so $\mu(A_{ij}) = 0$ for all $i, j \in \mathbb{N}_n, i \neq j$. Therefore, $\mu(B) = 0$. From this final contradiction, it is clear that the semi-discrete transport problem must be Monge under the shift characterization. \square

Corollary 3.3. *If the semi-discrete transport problem is defined as given in Section 2.2, and c is a p -norm for some $p \in (1, \infty)$, then Equation (2.15) is satisfied, and the optimal transport solution is unique μ -a.e.*

Proof. Suppose c is a p -norm for some $p \in (1, \infty)$, and assume the semi-discrete transport problem is characterized by shifts as given in Theorem 2.3. By the triangle inequality, for all $\mathbf{x} \in X, i, j \in \mathbb{N}_n, i \neq j$, $c(\mathbf{x}, \mathbf{y}_i) \leq c(\mathbf{x}, \mathbf{y}_j) + c(\mathbf{y}_i, \mathbf{y}_j)$. Hence, one consequence of the triangle inequality is that $g_{ij}(\mathbf{x}) \leq c(\mathbf{y}_i, \mathbf{y}_j)$ for all $\mathbf{x} \in A, i, j \in \mathbb{N}_n$ such that $i \neq j$. Therefore, $\{\mathbf{x} \in A \mid g_{ij}(\mathbf{x}) = k\} = \emptyset$ if $k < -c(\mathbf{y}_i, \mathbf{y}_j)$ or $k > c(\mathbf{y}_i, \mathbf{y}_j)$. This implies

$$\mu(\{\mathbf{x} \in A \mid g_{ij}(\mathbf{x}) = k\}) = 0 \quad \forall i, j \in \mathbb{N}_n, i \neq j, \quad \forall k \in (-\infty, -c(\mathbf{y}_i, \mathbf{y}_j)) \cup (c(\mathbf{y}_i, \mathbf{y}_j), \infty).$$

By Theorem 3.2, $\mu(B) = 0$. Thus, for any $i, j \in \mathbb{N}_n, i \neq j$, $\mu(A_{ij}) = 0$. Since the problem assumes nothing about the probability density ν , it must be the case that

$$\mu(\{\mathbf{x} \in A \mid g_{ij}(\mathbf{x}) = k\}) = 0 \quad \forall i, j \in \mathbb{N}_n, i \neq j, \quad \forall k \in [-c(\mathbf{y}_i, \mathbf{y}_j), c(\mathbf{y}_i, \mathbf{y}_j)].$$

Therefore, $\mu(\{\mathbf{x} \in A \mid g_{ij}(\mathbf{x}) = k\}) = 0$ for all $i, j \in \mathbb{N}_n, i \neq j$, and for all $k \in \mathbb{R}$, and uniqueness follows from Corollary 4 of [2]. \square

4. CONCLUSIONS

This paper resolves issues of partitioning and uniqueness for semi-discrete transport problems using a large class of ground cost functions: the p -norms. If the cost function is a p -norm with $p \in (1, \infty)$, the above arguments ensure that μ -a.e. unique solutions exist for semi-discrete transport problems. As the examples show, if the cost function is a p -norm with $p = 1$ or $p = \infty$, the solution may or may not constitute a μ -a.e. unique partition of the continuous space.

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