Topological Models for 3-configurations

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Abstract

Models of geometries are desirable for verifying properties of consistency, completeness, and independence of axiom systems. We consider 3-configurations of order v (for $3 \leq v \leq 100$) and replication number r, with necessarily $v \geq 2r+1$ and $3 \mid vr$. For each of the 1345 such pairs (v, r), we find a topological model for one corresponding 3-configuration, focusing on Steiner Triple Systems and partially balanced incomplete block designs where possible.

I. Introduction

A *3-configuration* is a finite geometry with v points and b lines, satisfying:

(i) every point is on exactly r lines;

(ii) every line contains exactly 3 points;

(iii) every pair of distinct points belong to at most one line.

The collineation graph for a geometry is called its *Menger graph*: vertices as points, with two vertices adjacent if their corresponding points are collinear. If this graph is complete, then we have a *Steiner Triple System* (STS), that is a $(v, b, r, 3, 1)$-BIBD, with $b = v(v-1)/6$ and $r = (v-1)/2$. (See [4] for amplifications of all terminology, notation, and background theory for this paper.) If the Menger graph is not complete, we have a $(v, b, r, 3; 0, 1)$-design ($\lambda_1 = 0$ for pairs of noncollinear points, $\lambda_2 = 1$ for collinear ones). If the incomplete (and non-empty) Menger graph is strongly regular, the design is a $(v, b, r, 3; 0, 1)$-PBIBD.

Models of geometries are desirable for verifying properties of consistency, completeness, and independence of axiom systems. There is also the appeal of representing an abstract mathematical system in a concrete manner. Necessary and sufficient conditions for the existence of a 3-configuration of order v and replication number $r \geq 1$ are:

a) $v \geq 2r + 1$ (since degree 2 at a vertex u is needed for each block on u);

b) $3 \mid vr$ (since in general $vr = bk$ and here $k = 3$).

For each of the 1345 such pairs (v, r) with $3 \leq v \leq 100$, we have found a topological model, and in this paper we outline our approach, with representative constructions. Our topological spaces will include surfaces (mostly orientable, but some not), pseudosurfaces, and generalized pseudosurfaces. Our constructions normally arise as covering spaces (possibly branched) over voltage graph imbeddings. (For background concerning voltage graphs, see [2] or Chapter 10 of [4].) The covering graphs will be Menger graphs of the relevant geometry, in each case being a Cayley graph for the voltage group Γ. The covering imbedding will have bichromatic dual (inherited from the voltage graph imbedding), with all the regions of one color class (white, say) being triangular; these b regions will model the lines of the geometry. The other (hyper-) regions will be what would remain in the ambient space, if the images of the points and lines had been removed. See Figures 1 and 2 for $(v, r) = (6, 2)$ and $(9, 2)$ respectively. The former imbeds $K_{3(2)}$ in S_0 and yields a $(6, 4, 2, 3; 0, 1)$-PBIBD; the latter puts $C_3 \times C_3$ on S_1 and gives a $(9, 6, 2, 3; 0, 1)$-PBIBD. In Figure 1, the voltage graph imbedding is index 2. (If we lift by $\mathbb{Z}_6$, we get two isomorphic copies of the covering imbedding; if we lift by $\mathbb{Z}_3 \cong \langle 2 \rangle$ we get just one.) In Figure 2, the voltage group is $\mathbb{Z}_3 \times \mathbb{Z}_3$, and the imbedding is index 1.

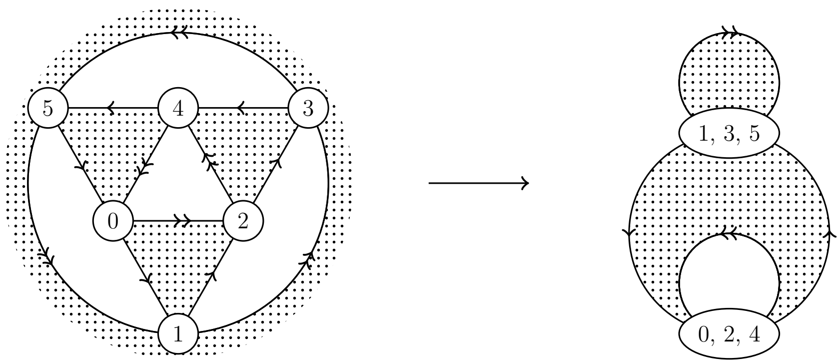


Figure 1: $(v, r) = (6, 2)$

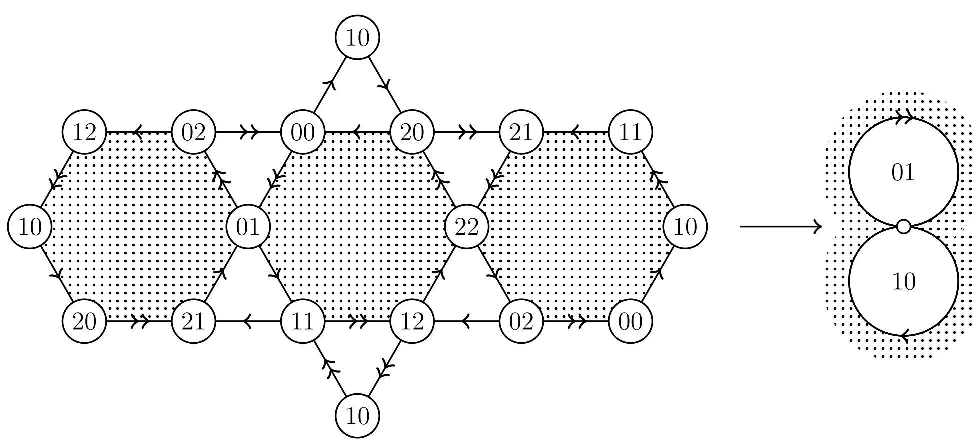


Figure 2: $(v, r) = (9, 2)$

For aesthetic purposes, as well as ease of construction, we prefer (in no particular order):

(1) orientable spaces to nonorientable ones;

(2) surfaces to pseudosurfaces, and pseudosurfaces to generalized pseudosurfaces;

(3) Menger graphs which are either complete ($v = 2r + 1$) or strongly regular; in both cases the block designs are more interesting.

(4) voltage groups which are Abelian (and especially the cyclic ones);

(5) covering imbeddings which have interesting symmetries (for example, those arising from index 1 voltage graphs are Cayley maps and hence the voltage group has a regular action—preserving the lines—on the vertices of the map);

(6) maximizing the Euler characteristic of the ambient space above; a sufficient but not necessary condition for this is that all the hyperregions be triangular.

Thus we find the two constructions of this section to be very satisfactory. For the rest of the paper we do not picture the covering imbeddings we construct, as the voltage graph imbeddings provide all the relevant information.

II. Small v

Several interesting geometries, and some famous ones, occur for $v \leq 10$. we have already seen two of these. In addition, we find:

A. Figure 3 (with $\Gamma = \mathbb{Z}_7$) imbeds K_7 on S_1, yielding a $(7, 7, 3, 3, 1)$-STS, called the *Fano plane*, and also known as the projective geometry $PG(2, 2)$. There are 21 symmetries. For topological models of $PG(2, n)$ in general, see [3]; but only for $n = 2$ do we get a 3-configuration.

B. Figure 3, but now with $\Gamma = \mathbb{Z}_8$, imbeds $K_{4(2)}$ on S_1, producing an $(8, 8, 3, 3; 0, 1)$-PBIBD. There are 24 symmetries.

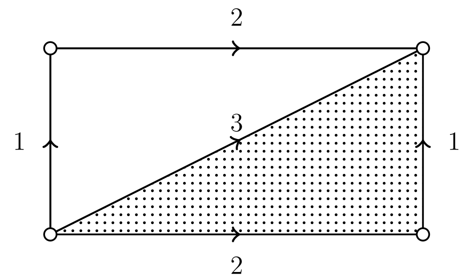


Figure 3: The Fano Plane, $PG(2,2)$

C. Figure 2, still using $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ but now augmented by a third loop carrying voltage $11 = (1,1)$, imbeds $K_{3(3)}$ on S_1, giving a $(9,9,3,3;0,1)$-PBIBD called the *geometry of Pappus*. Now all hyperregions are triangular, and there are 27 symmetries. In general, $K_{3(n)}$ is the strongly regular Menger graph for a $(3n, n^2, n, 3; 0, 1)$-PBIBD, with a topological model on $S_{(n-1)(n-2)/2}$ having $3n^2$ symmetries and maximum Euler characteristic. The index 2 voltage group for $n > 3$ is $\mathbb{Z}_n$, and routine surgery is employed (see p. 133 of [4]).

D. Now return to $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ and add a fourth loop (bearing voltage $12 = (1,2)$ to Figure 2 (retaining the third loop). The result is an imbedding of K_9 on S_7, a $(9,12,4,3,1)$-STS called the *geometry of Young* and also the affine plane $AG(2,3)$. The hyperregions are three dodecagons, which seem inefficient. But a compensating feature is that each loop lifts to a parallel class of lines partitioning the point set of the geometry. This makes the corresponding block design resolvable. We associate with loops bearing 10, 11, and 12 lines of slope 0, 1 and 2 respectively, The fourth loop 01 lifts to the three vertical lines.

E. An imbedding of Π' (the complement of the Petersen graph) on S_2 models the *geometry of Desargues*, a $(10,10,3,3;0,1)$-PBIBD. There are only 3 symmetries. Neither Π nor Π' is a Cayley graph, so no convenient voltage graph imbedding exists. See [1] for an ad hoc construction. The graph Π' is also the line graph $L(K_5)$; $L(K_3) = C_3$ and $L(K_4) = K_{3(2)}$; see Figure 1. In general, $L(K_n)$ is strongly regular and serves as the Menger graph for the 3-configuration derived by taking the edges of K_n as points and the triangles (3-cycles) of K_n as lines. See [5] for a topological study of these

$\left(\binom{n}{2}, \binom{n}{3}, (n-2), 3; 0, 1\right)$-PBIBDs.

Another useful class of strongly regular graphs consists of the regular complete m-partite graphs $K_{m(n)}$, as seen in Figure 1, **B.** and **C.** above, and in Sections IV. and V. below.

III. Small $\boldsymbol{r}$

We readily find general constructions for the first four values of r.

A. For $\boldsymbol{r = 1}$, $v = 3k$ for some $k \in \mathbb{N}$ (since $3 \mid vr$) and kC_3 on kS_0 models the $(3k, k, 1, 3; 0, 1)$-PBIBD (for $k > 1$, but a BIBD for $k = 1$). Note that this topological space has k components, and that all the hyperregions are triangular.

B1. For $\boldsymbol{r = 2}$ and $v = 6k$, $kK_{3(2)}$ on kS_0 gives a $(6k, 4k, 2, 3; 0, 1)$-BD (neither a BIBD nor a PBID for $k > 1$, but a PBIBD for $k = 1$). See Figure 1 for $k = 1$.

B2. For $\boldsymbol{r = 2}$ and $v = 6k+3$, $((k-1)K_{3(2)}$ on $(k-1)S_0) \cup (C_3 \times C_3$ on $S_1)$ gives a $(6k+3, 4k+2, 2, 3; 0, 1)$-BD (except a PBIBD for $k = 1$; see Figure 2).

C. For $\boldsymbol{r = 3}$ and $v = 7$, Figure 3 suffices, with $\Gamma = \mathbb{Z}_7$. But Figure 3 satisfies the Kirchoff Voltage Law (KVL; see [4]) in $\mathbb{Z}_\infty$ (that is, in $\mathbb{Z}_v$, for all $v \geq 7$). Thus for $r = 3$ and $v \geq 7$, we immediately obtain a model for a $(v, v, 3, 3; 0, 1)$-BD.

D. For $\boldsymbol{r = 4}$ and $v = 3k$ ($k \geq 4$; see II.**D.** for $k = 3$), we augment Figure 3 as shown in Figure 4; the unshaded triangle lifts to $3k$ lines above, while the loop lifts to k lines (all triangles), using $\Gamma = \mathbb{Z}_{3k}$. The result is a Cayley graph for $\mathbb{Z}_{3k}$, generated by $\Delta = \{1, 2, 3, k\}$, imbedded in S_{2k+1} and modeling a $(3k, 4k, 4, 3, 1)$-BD. There are k hyperregions, all dodecagons.

E. For $\boldsymbol{r \equiv 2 \pmod 3}$, $\boldsymbol{r \geq 5}$, we will need either a non-cyclic voltage group or a higher-index voltage graph. See Section IV.**B.(iii)b)** and **a)** respectively.

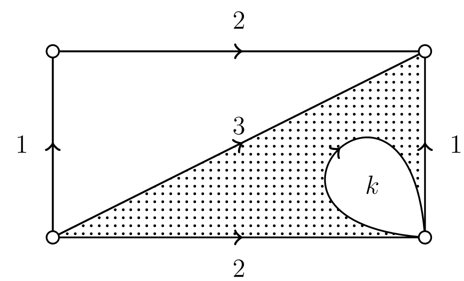


Figure 4: Cayley graph for $\mathbb{Z}_{3k}$

IV. The General Constructions

Recall that $3 \mid vr$. We split into two cases.

A. $3 \mid r$. We choose r generators for $\Gamma = \mathbb{Z}_v$ that we can partition into KVL triples, each triple bounding a triangular 2-cell, and then assemble these into an appropriate topological space. For example, suppose we want a 3-configuration with $v = 21$ and $r = 9$. We start with $K_{7(3)}$, a strongly regular Menger graph. We want a suitable Cayley map for $K_{7(3)}$. Then we must choose between $\mathbb{Z}_{21}$ and the semi-direct product of $\mathbb{Z}_7$ with $\mathbb{Z}_3$ as our voltage group. The former is abelian, with one element 7 (and its inverse) of order 3, while the latter is non-abelian and has seven pairs of elements of order 3. Elements of order 3 are useful in producing triangles (as in Figure 4), but the abelian property is irrestible for ease of computation (and, as it turns out, one element of order 3 is all we need here)—so we try $\Gamma = \mathbb{Z}_{21}$. Choosing $\Delta = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$, we find $G_\Delta(\Gamma) = K_{7(3)}$. We seek to partition Δ into three KVL triples, and as Δ consists of four odd numbers (o) and five even ones (e), we look for equations of the form $o + o - e = 0$, $o + o - e = 0$; $e + e - e = 0$. (As we will see later, it is nice to avoid the form $a + b + c = 21$.) This works: $1 + 9 - 10 = 0$; $3 + 5 - 8 = 0$; $2 + 4 - 6 = 0$. These equations yield three white triangles for our voltage graph imbedding. Now we want three black KVL triangles (for preference (6) of Section I.), with each generator (still from Δ) having the opposite sign to that already used (for preference (1)); preferences (3) and (4) have already been attended to; preferences (2) and (5) lie ahead.) We cannot find them. But if we rewrite $3 + 5 - 8 = 0$ as $-3 - 5 + 8 = 0$, we can: $-2 - 8 + 10 = 0$; $-4 - 1 + 5 = 0$; $6 - 9 + 3 = 0$. We use these six 3-cycles (three white, three black) to bound six triangular regions and then identify edges around the peripheries to obtain the KVL voltage graph imbedding (in S_2) similar to that of Figure 5. For preferences (2) and (5), we check that our voltage graph has just one vertex (x). It doesn't (there are three). As it stands, our voltage graph is imbedded in the pseudosurface $S(1;1(3))$—a sphere on which one set of three vertices have been identified. This leads to a covering imbedding in $S(1;21(3))$ for $K_{7(3)}$. But if we change $-2-8+10=0$ to $-2+10-8=0$ (with our alternative nonabelian voltage group, such a maneuver might not be possible), we get everything we want—including preferences (2) and (5). This is displayed in Figure 5.

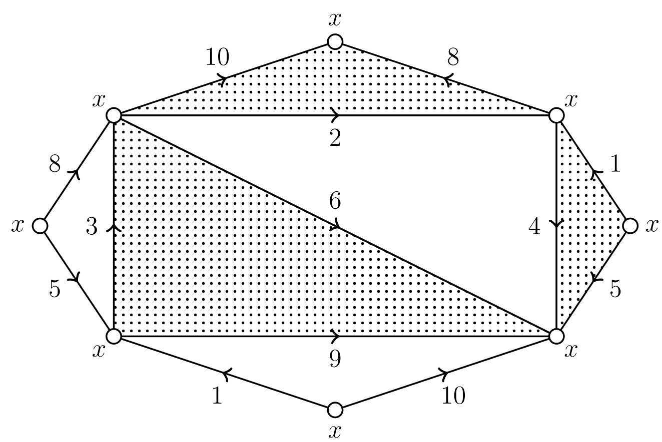


Figure 5: Cayley graph of $\mathbb{Z}_{21}$ satisfying preferences (1)–(6)

The lift will be a bichromatic–dual imbedding of $K_{7(3)}$ in S_{22} ($p=21$, $q=189$, $r=r_3=6\times 21=126$), modeling a 3-configuration which is a $(21,63,9,3;0,1)$-PBIBD. In Table 1 we display the lines of this geometry, each 21 of which are cyclically generated by the white region they project down onto.

We have two surprises in store. Firstly, since the KVL holds in $\mathbb{Z}_\infty$, we have an infinite class of orientable surface models for 3-configurations having $v\geq 21$, $r=9$, and at least v translational symmetries. (We are not likely to have strong regularity, for $v>21$.) The covering surface is S_{v+1}, as $p=v$, $q=9v$, and $r=r_3=6v$. Secondly, if we modify Δ to $\Delta' = \Delta\cup\{7\}$, then $G_{\Delta'}(\mathbb{Z}_{21})=K_{21}$. Now add a loop carrying voltage 7 inside any one of the three back triangles of Figure 5. Color the loop region white; it will

0	2	6	0	1	10	0	8	5
1	3	7	1	2	11	1	9	6
2	4	8	2	3	12	2	10	7
	⋮			⋮			⋮	
20	1	5	20	0	9	20	7	4

Table 1: The lines of our geometry for $K_{7(3)}$ imbedded in S_{22}

lift to seven more white triangles. The result is an imbedding of K_{21} on S_{36}, modeling STS(21) with at least 21 symmetries. This almost certainly does not maximize Euler characteristic for this $(21, 70, 10, 3, 1)$-BIBD, as the modified back triangle now lifts to seven dodecagons.

We note that, for $v > 21$, the loop region no longer lifts to triangles.

B. $\boldsymbol{3 \mid v}$. Again we split, this time into three cases.

(i) If $r \equiv 0 \pmod 3$, proceed as in A, avoiding the use of k as a generator (where $\Gamma = \mathbb{Z}_{3k}$, with $v = 3k$).

(ii) If $r \equiv 1 \pmod 3$, add a loop carrying voltage k to a voltage graph imbedding, as in (i). See Figure 4, for example.

(iii) If $r \equiv 2 \pmod 3$, we have more of a challenge, since in $\Gamma = \mathbb{Z}_{3k}$ we have only one element (and its inverse) of order 3; thus there are no more loops to add. We split into two cases.

a) $v = 3k \pm 3$. We stick with $\mathbb{Z}_v$, but use an index 3 voltage graph. For example, see Figure 6, where $\Gamma = \mathbb{Z}_{24}$, for the case $v = 24$, $r = 5$. We lift by $\langle 3 \rangle \approx \mathbb{Z}_8$. The design is a $(24, 40, 5, 3; 0, 1)$-BD, modelled on S_{23}.

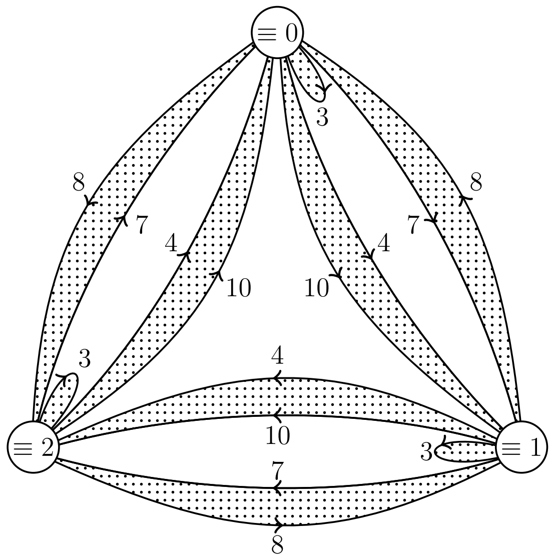


Figure 6: Cayley graph of $\mathbb{Z}_{24}$

For $r = 8$, augment Figure 6 by Figure 7 (three times). The new surface is S_{38}, carrying a $(24, 64, 8, 3; 0, 1)$-BD. Finally, for $r = 11$, augment the voltage graph imbedding for $r = 8$ by Figure 8 (three times). The final graph is $K_{12(2)} = G_{\Delta}(\mathbb{Z}_{24})$, with $\Delta = \{1, 2, 3, \cdots, 11\}$. The final surface is S_{53}, carrying a $(24, 88, 11, 3; 0, 1)$-PBIBD. This imbedding is reasonably efficient, as the hyperregions are: $r_4 = 30$, $r_8 = 18$.

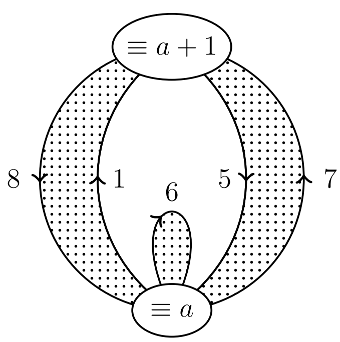


Figure 7: Augmentation for $r = 8$

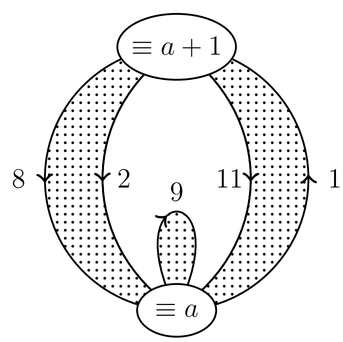
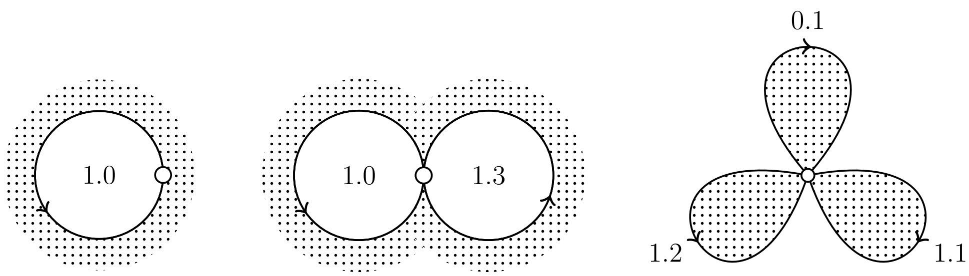


Figure 8: Augmentation for $r = 11$

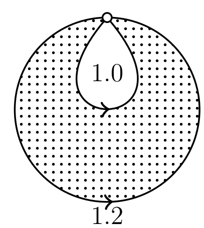
This completes the case $v = 24$, as the more routine r values are treated as in (i) and (ii).

b) $9 \mid v$, so $v = 9k$. We use $\Gamma = C_3 \times C_{3k}$, so that we have four elements of order 3 (and their inverses): $(1,0)$, $(1,k)$, $(1,2k)$, and $(0,k)$. (As before, we abbreviate: 1.0, 1.k, 1.2k, and 0.k.) We illustrate an iterated construction that will cover all possible r values for $v = 27$ (so $k = 3$): $1 \leq r \leq 13$.

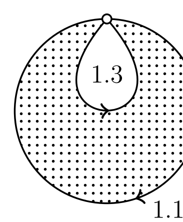


$r = 1$ $\qquad$ $r = 2$ $\qquad$ $r = 3$

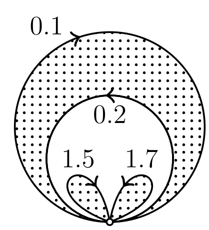
For $r = 4$, augment the voltage graph of $r = 3$ by:



For $r = 5$, augment the voltage graph of $r = 4$ by:



For $r = 6$, augment the voltage graph of $r = 3$ by:



For $r = 7$ and 8, augment the voltage graph for $r = 6$ as done for $r = 4$ and 5 above.

So far, we have used KVL triples $0.1 + 1.1 = 1.2$ and $0.2 + 1.5 = 1.7$. Continue, using $0.4 + 1.4 = 1.8$ and $0.3 + 1.3 = 1.6$, finishing with one more augmentation as for $r = 4$. This gives an imbedding of K_{27} on S_{79} and a $(27, 117, 13, 3, 1)$-STS.

V. A Stubborn Situation

There is one situation that resists the approaches outlined above. Suppose, for example, we want to imbed $K_{10(2)}$ so as to model a $(20, 60, 9, 3; 0, 1)$-PBIBD using $\Gamma = \mathbb{Z}_{20}$. Then our generating set is $\Delta = \{1, 2, 3, \cdots, 9\}$, which contains five odd generators. At least one of our three KVL triples must involve an odd number of these odd generators, but no even order cyclic group can have such a triple.

However, in IV.**A.** we modeled STS(21) on S_{36}. Now, if we start with a 3-configuration model of STS$(v + 1)$ on a topological space T, and remove one object (point of the geometry) and all the $r = v/2$ blocks incident with that object from T, what remains on T is a model of the partially balanced incomplete block design with strongly regular Menger graph $K_{(v/2)(2)}$. In the present case, we have $K_{10(2)}$ on S_{36}. The imbedding might not be very efficient, as there will be a large region (at least a 30-gon) where the deleted point was. But we do have a topological model on the parameters of concern.

In general, this situation will occur for v even and $r = 3k$ (for $k \in \mathbb{N}$), so that $v = 2r + 2 = 6k + 2$. Thus $v + 1 \equiv 3 \pmod 6$, and there does exist an STS$(v+1)$. Since $3 \mid r$, we can model the STS$(v+1)$ as in IV.**B.(i)** and **(ii)**. Thus the deletion procedure of this section will apply. The key is that, if v is even (so that we cannot accommodate an odd number of odd generators in KVL triples, as in the example of this section), then $v + 1$ is odd and this impediment disappears.

This completes the outline and illustration of the program worked upon sporadically by the second author over a period of fifteen years, and contributed to significantly by the first author during the final year.

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